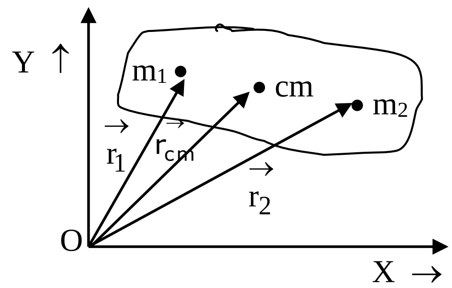
CHAPTER 07 – SYSTEM OF PARTICLES AND ROTATIONAL MOTION

Centre of Mass

1) **Equations for centre of mass**



The position vector of the centre of mass of the body is given by

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum m\vec{r}}{M}$$

M is the mass of the body

$m_1, m_2, \ldots\ldots m_n$ are the masses of n particles of the rigid body having position vectors $\vec{r}_1, \vec{r}_2, \ldots\ldots \vec{r}_n$ respectively w.r.t origin O.

In coordinate system, $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, $\vec{r}_3 = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$……

Then x, y and z coordinates of the position vector r_{CM} of the centre of mass is given by:

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots + m_nx_n}{m_1 + m_2 + m_3 + \ldots + m_n} = \frac{\sum mx}{M}$$; Similarly $y_{CM} = \frac{\sum my}{M}$; $z_{CM} = \frac{\sum mz}{M}$

2) **Centre of mass for two particles system:**

Consider two particles of mass m_1 and m_2 having position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively w.r.t origin O.

The position vector of the centre of mass of the system is $\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

In coordinate system, $\vec{r}_1 = x_1\hat{i} + y_1\hat{j}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j}$

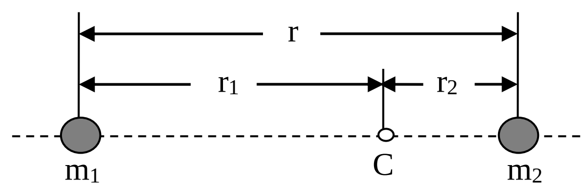
Then x and y coordinates of the position vector $\vec{r}_{cm}$ of the centre of mass is given by $x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$

and $y_{CM} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$

3) For two particles of equal mass the centre of mass lies exactly midway between them.

4) Centre of mass of two particles, lies on the line joining them.
If particles are of unequal masses then it lies near heavier particle.
m_1 and m_2 ($m_2 > m_1$) are two particles separated by a distance r. Let the distances of these particles from the centre of mass be r_1 and r_2.

$r_1 = \frac{m_2 r}{m_1 + m_2}$ and $r_2 = \frac{m_1 r}{m_1 + m_2}$



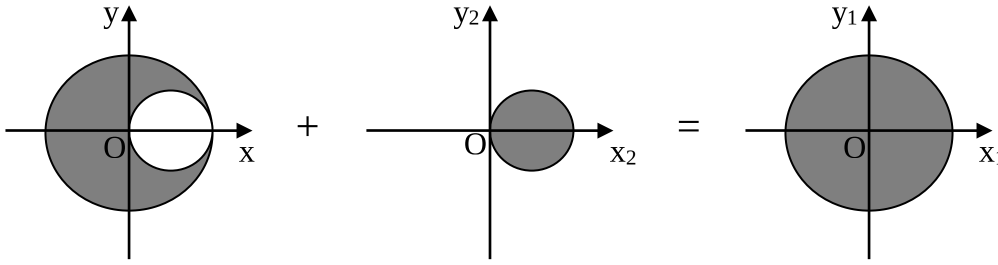
5) Velocity of centre of mass: $\vec{v}_{cm} = \frac{d\vec{r}_{CM}}{dt} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$

6) Acceleration of centre of mass: $\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2}$

7) **Problem solving tricks for centre of mass concept:**

- Make use of the symmetry of the object, be it point line or plane.
- If the object can be divided into several parts, treat each part as a particle, located at its own centre of mass
- Choose the axis wisely: If given system is a group of particles, choose one of the particles as origin. If system is a body with a line of symmetry, consider it as x-axis. The choice of origin is completely arbitrary; the location of centre of mass is same regardless of the origin from which it is measured.

8) **Centre of mass in truncated configuration:**



Consider a body of mass M and let m be the mass of the removed portion.
If a portion of a body is taken out, the remaining portion may be considered as [original mass (M) - mass of the removed part (m)]
= {Original mass (M)} + {- mass of the removed part (m)}

The formula changes to:

$x_{CM} = \frac{Mx_1 - mx_2}{M - m}$ and $y_{CM} = \frac{My_1 - my_2}{M - m}$

where (x_1, y_1) are the coordinates of the centre of mass of the original mass.
where (x_2, y_2) are the coordinates of the centre of mass of the removed portion.

Rotational Motion

1) Linear displacement = radius × angular displacement, i.e. $s = r\theta$
2) Linear velocity = radius × angular velocity i.e. $v = r\omega$
3) If $d\omega$ is the change in angular velocity in a time dt then instantaneous angular acceleration $\alpha = \frac{d\omega}{dt}$

   Also $\alpha = \frac{\omega_2 - \omega_1}{t}$
4) Linear acceleration = radius × angular acceleration i.e. $a = r\alpha$
5) **Equations of angular motion:**

   $\omega_2 = \omega_1 + \alpha t$ ω_1 = Initial angular velocity
   $\omega_2^2 = \omega_1^2 + 2\alpha\theta$ ω_2 = Final angular velocity
   $\theta = \omega_1 t + ½ \alpha t^2$ θ - Angular displacement
6) Moment of inertia about any axis of rotation is defined as $I = \Sigma mr^2$ where m is the mass of the particle of the body r is its perpendicular distance from the axis, the summation being extended for every particle of the body.
7) The moment of inertia of a body about a given axis can also be written as $I = Mk^2$, M is mass of the body, k is a distance called the radius of gyration of the body about the given axis.

   $k = \sqrt{\frac{I}{M}}$
8) M.I of a rigid body is given by $I = \int r^2 dm$ where dm is mass of a particle at a distance r from the axis of rotation.
9) Radius of gyration k of a body about an axis is equal to the root mean square distances of the particles from the axis of rotation.

   $k = \sqrt{\frac{m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots + m_nr_n^2}{m_1 + m_2 + m_3 + \ldots + m_n}}$

   If particles are of same mass, then $k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}}$
10) For asymmetrical separation radius of gyration will change $k = \sqrt{\frac{I - I_d}{M - m}}$

    m = detached mass, I_d = M.I of detached mass w.r.t same axis
11) For asymmetrical attachment radius of gyration will change $k = \sqrt{\frac{I + I_d}{M + m}}$

    m = added mass, I_d = M.I of attached mass w.r.t same axis
12) **Moment of inertia of some of the bodies**

Body	Axis of rotation	Moment of inertia
Circular ring of mass M and radius R	About an axis passing through the centre and perpendicular to the plane of the ring	$I = MR^2$
	About diameter	$I = \frac{MR^2}{2}$
	About tangent (Parallel to diameter)	$I = \frac{3MR^2}{2}$
	About a tangent perpendicular to its plane	$I = 2MR^2$
Circular disc of mass M and radius R	About an axis passing through the centre and perpendicular to the plane of the ring	$I = \frac{MR^2}{2}$
	About diameter	$I = \frac{MR^2}{4}$
	About tangent (Parallel to diameter)	$I = \frac{5MR^2}{4}$
	About a tangent perpendicular to its plane	$I = \frac{3MR^2}{2}$
Solid sphere of mass M and radius R	About diameter	$I = \frac{2MR^2}{5}$
	About tangential axis	$I = \frac{7MR^2}{5}$
Hollow sphere of mass M and radius R	About diameter	$I = \frac{2MR^2}{3}$
	About tangential axis	$I = \frac{5MR^2}{3}$
Solid cylinder of mass M, radius R	About its axis	$I = \frac{MR^2}{2}$

and length ℓ	About an axis passing through the centre and perpendicular to its length.	$I = \frac{MR^2}{4} + \frac{M\ell^2}{12}$
Hollow cylinder of mass M, radius R and length ℓ	About its axis	$I = MR^2$
	About an axis passing through the centre and perpendicular to its length.	$I = \frac{MR^2}{2} + \frac{M\ell^2}{12}$
Long thin rod of mass m and length ℓ	Passing through the centre and perpendicular to the length	$I = \frac{M\ell^2}{12}$
	Passing through one end and perpendicular to the length	$I = \frac{M\ell^2}{3}$
Rectangular lamina of mass M, length L and breadth B	Passing through the centre and perpendicular to the plane	$I = M\left[\frac{L^2 + B^2}{12}\right]$
	Passing through the centre and perpendicular to breadth in its plane	$I = M (B^2/12)$
	Passing through the centre and perpendicular to the length in its plane	$I = M(L^2/12)$

13) Kinetic energy of rotation of a body $K_R = ½I\omega^2 = \frac{L^2}{2I} = \frac{1}{2}L\omega$

14) Total kinetic energy of a rolling body $K = K_T + K_R = ½\,mv^2 + ½I\omega^2$
L – Angular momentum, I – moment of inertia.

15) Angular momentum $L = mvr = I\omega$

Also $\vec{L} = \vec{r} \times \vec{p}$ or $L = rp\sin\theta$

where m = mass, v = velocity, I = moment of inertia, ω = angular velocity, p = linear momentum, r = distance between point of application of force and axis of rotation.

16) Angular momentum $L = \frac{2K}{\omega} = \sqrt{2IK}$

K – Rotational kinetic energy, ω - angular velocity

17) According to the law of conservation of angular momentum, $L = I\omega$ = constant
i.e $I_1\omega_1 = I_2\omega_2$

18) Torque or moment of force $\vec{\tau} = \vec{r} \times \vec{F}$
Or $\tau = I\alpha = rF\sin\theta$ where θ is the angle between r and F

19) Anticlockwise torque is taken as positive and clockwise torque is taken as negative.

20) Torque is also equal to rate of change of angular momentum i.e $\tau = \frac{dL}{dt}$ (Newton's 2nd law of rotatory motion)

21) If the torque remains constant then work done by torque is $W = \tau\,\Delta\theta$
Δθ is the angular displacement

Work done by the torque during an angular displacement from θ_1 to θ_2 is $W = \int_{\theta_1}^{\theta_2} \tau\, d\theta$

22) Power of a torque is $P = \frac{dW}{dt} = \tau\,\omega$

23) Angular impulse is the product of torque and time for which the torque acts. It is also equal to the change in angular momentum.

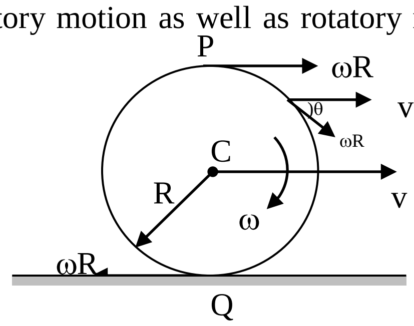
24) **Comparison between linear motion and rotatory motion**

Linear Motion	**Rotational Motion**
If v is constant, displacement s = vt	If ω is constant angular displacement $\theta = \omega t$
Velocity $v = \frac{ds}{dt} = r\omega$	Angular velocity $\omega = \frac{d\theta}{dt} = \frac{v}{r}$
Linear acceleration $a = \frac{dv}{dt} = v\frac{dv}{ds} = r\alpha$	Angular acceleration $\alpha = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} = \frac{a}{r}$
Mass (m)	Moment of inertia $I = \sum mr^2 = Mk^2$
Linear momentum $p = mv = \sqrt{2mK_T}$	Angular momentum $L = I\omega = \sqrt{2IK_R}$
Force $F = ma = \frac{dp}{dt}$	Torque $\tau = I\alpha = \frac{dL}{dt}$
Work $W = Fs = ½\,m(v^2 – u^2)$	Work $W = \tau\,\theta = ½I(\omega_2^2 – \omega_1^2)$
Power $P = \vec{F}.\vec{v}$	Power $P = \vec{\tau}.\vec{\omega}$
Kinetic energy $K_T = ½\,mv^2 = \frac{p^2}{2m} = ½\,pv$	Rotational K.E $K_R = ½\,I\omega^2 = \frac{L^2}{2I} = ½L\omega$
Equations of translational motion: $v = u + at$	**Equations of rotational motion:** $\omega_2 = \omega_1 + \alpha t$

$a = \frac{v-u}{t}$ $s = ut + \frac{1}{2} at^2$ $v^2 = u^2 + 2as$ $s = \left(\frac{u+v}{2}\right)t$	$\alpha = \frac{\omega_2 - \omega_1}{t}$ $\theta = \omega_1 t + \frac{1}{2} \alpha t^2$ $\omega_2^2 = \omega_1^2 + 2\alpha\theta$ $\theta = \left(\frac{\omega_1 + \omega_2}{2}\right)t$

Rolling motion

1) When a body performs translatory motion as well as rotatory motion together then it is said to undergo rolling motion.



$v = v_{cm}$ = translational speed of the centre of mass

R = radius of the object, ω = angular speed

2) **Rolling Kinetic energy**

M = mass of the body, v_{cm} = velocity of the centre of mass (velocity of the rolling body), k = radius of gyration, R = radius of the body

a) $K_{tot} = K_T + K_R = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv_{cm}^2\left[\frac{k^2}{R^2}+1\right]$

b) $K_T : K_R : K_{tot} = 1 : \left[\frac{k^2}{R^2}\right] : \left[\frac{k^2}{R^2}+1\right]$

3) **Rolling of a body on a smooth inclined plane:**

h – height of the incline, ℓ - length of the incline, θ - angle of inclination, k – radius of gyration. R – Radius of rolling body.

For pure sliding

Physical quantities	Pure rolling	Pure sliding ($k^2/R^2=0$)
Acceleration	$a_R = \frac{g\sin\theta}{1+\frac{k^2}{R^2}}$	$a_s = g\sin\theta$
Velocity at the bottom of incline	$v_R = \sqrt{\frac{2gh}{1+\frac{k^2}{R^2}}}$	$v_s = \sqrt{2gh} = \sqrt{2g\ell\sin\theta}$
Time taken to reach the bottom of incline of length ℓ	$t_R = \sqrt{\frac{2\ell\left(1+\frac{k^2}{R^2}\right)}{g\sin\theta}}$	$t_s = \sqrt{\frac{2\ell}{g\sin\theta}}$

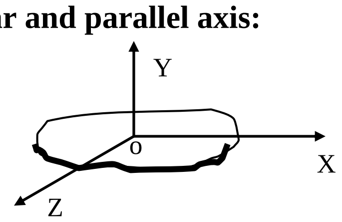
4) Time taken by rolling body to reach the bottom, $t_R = \sqrt{\frac{2\ell\left(1+\frac{k^2}{R^2}\right)}{g\sin\theta}}$

When different bodies are allowed to roll down an inclined plane then the body with

(i) least $\frac{k^2}{R^2}$ will reach first (ii) highest $\frac{k^2}{R^2}$ will reach last

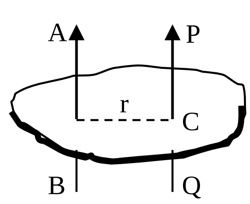
Theorem of perpendicular and parallel axis:

1)



If I_X, I_Y and I_Z represent the moment of inertia of the body about the axes OX, OY and OZ respectively then according to perpendicular axes theorem $I_X + I_Z = I_Y$

2)



Let I_C be the moment of inertia of a body of mass M about an axis (PQ) passing through the centre of gravity C.
According to the theorem of parallel axes moment of inertia of a body about an axis AB is
$I = I_C + Mr^2$
where r is the distance between the two axes AB and PQ