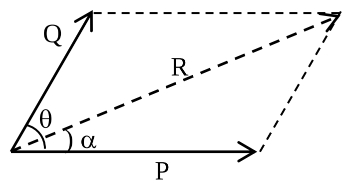
CHAPTER 04 - MOTION IN A PLANE

VECTORS

1) Unit vector $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$

2) **Magnitude of addition of two vectors:**



Resultant of addition of two vectors P and Q making an angle of θ is R = $\sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

Direction of the resultant force $\alpha = \tan^{-1}\left(\dfrac{Q\sin\theta}{P + Q\cos\theta}\right)$ where α is the angle between R and P

θ	R	α or Direction of resultant
0°	$R_{max} = P + Q$	0° i.e. along given vectors
90°	$R = \sqrt{P^2 + Q^2}$	$\alpha = \tan^{-1}\left(\frac{Q}{P}\right)$
180°	$R_{min} = P - Q$ (if P > Q	0° or 180° i.e. along greater vector

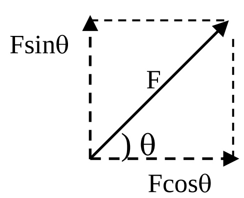
3) Resultant or addition of two identical vectors P making an angle θ between them is R = $2P\cos\left(\dfrac{\theta}{2}\right)$

R makes an angle of θ/2 with either vector.

4) When P = Q

θ	R	α
0°	2P	0°
60°	$\sqrt{3}$ P	30°
90°	$\sqrt{2}$ P	45°
120°	P	60°
180°	0	–

5) **Resolution of a vector:**



If F is a vector, which makes an angle θ with the x-axis then the component of F along x-axis is F_x = Fcosθ and the component of F along y-axis is F_y = Fsinθ also F = $\sqrt{F_x^2 + F_y^2}$

$\tan\theta = \dfrac{F_y}{F_x}$

6) **Representation of vector in one, two and three dimensional motion**

A vector in one dimensional motion along x- axis is given by $\vec{A} = x\hat{i}$ or $|\vec{A}| = x$

A vector in two dimensional motion in x-y plane is given by $\vec{A} = x\hat{i} + y\hat{j}$

$|\vec{A}| = \sqrt{x^2 + y^2}$

A vector in three dimensional motion is given by $\vec{A} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{A}| = \sqrt{x^2 + y^2 + z^2}$

where $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along x, y and z axis respectively. Where x, y and z are its components along three axes respectively. Also $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$

7) Velocity of a body in two dimensions $\vec{v} = \vec{v}_x + \vec{v}_y = \dfrac{(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}}{t_2 - t_1}$

8) Dot product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \cdot \vec{B}$

$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$ where θ is the angle between two vectors.

9) $|\vec{A} \cdot \vec{B}| = a_1b_1 + a_2b_2 + a_3b_3$ where $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along x, y and z axis.

$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and $|\vec{B}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

10) $\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos\theta = 1 \times 1 \times \cos 0 = 1 \because \theta = 0^o$

Similarly $\hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \because \theta = 90^o$

11) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ (Dot product is commutative)

12) Cross product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \times \vec{B}$

$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\sin\theta\,(\hat{n})$ where θ is the angle between two vectors and $\hat{n}$ is the unit vector.

13) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

14) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$

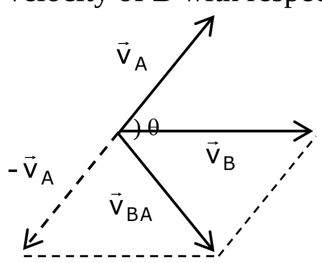
$\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j}$

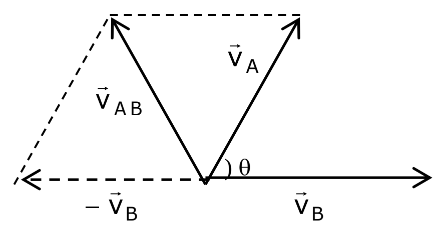
15) $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

16) **Relative velocity of two bodies moving in different directions:**

Relative velocity of A with respect to B is $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$

Relative velocity of B with respect to A is $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$





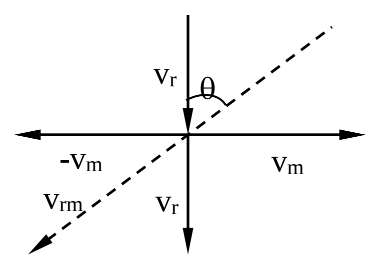
$|\vec{v}_{AB}| = |\vec{v}_A - \vec{v}_B| = \sqrt{v_A^2 + v_B^2 - 2v_Av_B\cos\theta}$

$|\vec{v}_{BA}| = |\vec{v}_B - \vec{v}_A| = \sqrt{v_A^2 + v_B^2 - 2v_Av_B\cos\theta}$

17) The angle through which man should hold umbrella with vertical to protect himself from rain which is falling vertically downwards is $\tan\theta = \frac{v_m}{v_r}$

Relative velocity of rain w.r.t man is $v_{rm} = v_r - v_m$

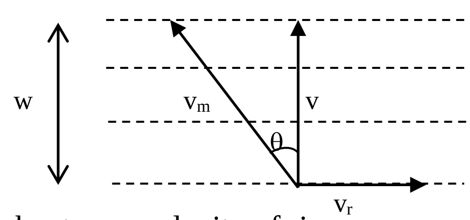
v_r = velocity of rain
v_m = velocity of man



18) When a man tends to cross a river along a shortest path, he should swim upstream making an angle θ given by $\sin\theta = \frac{v_r}{v_m}$

Time taken to cross the river over the shortest distance is

$t = \frac{w}{\sqrt{v_m^2 - v_r^2}} = \frac{w}{v}$



where resultant velocity $v = \sqrt{v_m^2 - v_r^2}$ v_m – velocity of man or boat; v_r – velocity of river

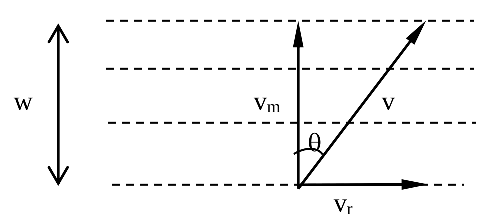
19) **When a man or a boat tends to cross the river in shortest time.**

Time taken to cross the river of width w in shortest possible time is $t = \frac{w}{v_m}$

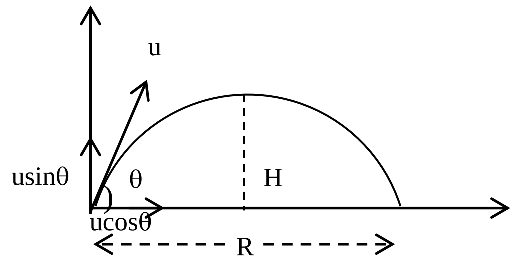
Resultant velocity $v = \sqrt{v_m^2 + v_r^2}$

$\tan\theta = \frac{v_r}{v_m}$

w – width of the river
v_m – velocity of man or boat
v_r – velocity of river water



OBLIQUE PROJECTILE MOTION



1)

	Along horizontal (x-axis)	Along vertical (y-axis)
Initial velocity	$u_x = u\cos\theta$	$u_y = u\sin\theta$
Acceleration	$a_x = 0$	$a_y = -g$
Velocity at maximum height	$v_x = u\cos\theta$	$v_y = 0$
Velocity at any instant t	$v_x = u\cos\theta$	$v_y = u_y - gt = u\sin\theta - gt$
Displacement	$x = u_x t = (u\cos\theta)t$	$y = u_y t - \frac{1}{2} gt^2$ $y = (u\sin\theta)t - \frac{1}{2}gt^2$

2) **Energy:**

When a projectile moves upward its kinetic energy decreases, potential energy increases but the total energy always remains constant.

- Initial kinetic energy $K = \frac{1}{2} mu^2$
- Initial potential energy $U = 0$

At the highest point:

- Kinetic energy $K' = \frac{1}{2}m (u\cos\theta)^2 = K \cos^2\theta$
- Potential energy $U = mgH = \frac{1}{2} mu^2 \sin^2\theta = K \sin^2\theta$
- Total energy $E = K + U = \frac{1}{2} mu^2$ = energy at the point of projection.

3) **Instantaneous velocity:**

- Vertical velocity $v_y = u_y - gt = u\sin\theta - gt$
- Horizontal velocity $v_x = u_x = u\cos\theta$
- The resultant velocity $\vec{v} = v_x\hat{i} + v_y\hat{j}$ or $v = \sqrt{v_x^2 + v_y^2}$
- The direction of the velocity w.r.t horizontal is $\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

4) **Velocity at a given height:**

- Vertical velocity $v_y = \sqrt{u_y^2 - 2gh} = \sqrt{u^2\sin^2\theta - 2gh}$
- Horizontal velocity $v_x = u_x = u\cos\theta$
- The resultant velocity $\vec{v} = v_x\hat{i} + v_y\hat{j}$ or $v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 - 2gh}$
- The direction of the velocity w.r.t horizontal is $\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

5) **Angular momentum:**

Angular momentum of the projectile at highest point of trajectory about the point of projection:

$$L = mvr = mvH = m(u\cos\theta)\left(\frac{u^2\sin^2\theta}{2g}\right) = \frac{mu^3\cos\theta\sin^2\theta}{2g}$$

6) **Equation of trajectory:**

- $y = ax - bx^2 = x\tan\theta\left(1 - \frac{x}{R}\right)$ where $a = \tan\theta$ and $b = \left(\frac{g}{2u^2\cos^2\theta}\right)$ are constant
- Horizontal range $R = \frac{a}{b} = \frac{u^2\sin 2\theta}{g}$

7) **Time of Flight:**

- Time of ascent = Time of descent $t = \frac{T}{2} = \frac{u\sin\theta}{g}$
- $T = \frac{2u_y}{g} = \frac{2u\sin\theta}{g}$

8) **Maximum Height:**

- $H = \frac{u_y^2}{2g} = \frac{u^2\sin^2\theta}{2g}$
- For maximum height, body should be thrown vertically upward ($\theta = 90^o$)

  $H_{max} = \frac{u^2}{2g}$

9) Time of flight and maximum height H depends only on u_y

If two projectiles thrown in different directions have equal time of flight then their initial vertical velocities are same so that their maximum height is also same.

10) **Range:**

- $R = (u\cos\theta)T = \frac{2u_x u_y}{g} = \frac{u^2 \sin 2\theta}{g}$

Maximum range:

- For a given velocity of projection the range is maximum (R_{max}), if the angle of projection is 45°.
- $R_{max} = \frac{u^2}{g} = 4H$
- Maximum height $H = \frac{u^2}{4g} = \frac{R_{max}}{4}$
- Time of flight $T = \sqrt{\frac{2R_{max}}{g}}$

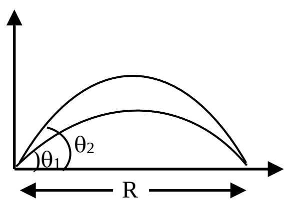
11) **Relation between range and maximum height:**

- $R = 4H\cot\theta$
- If $R = nH$, $\theta = \tan^{-1}\left(\frac{4}{n}\right)$

  If $R = H$, $\theta = \tan^{-1}(4) = 76°$

  If $R = 4H$, $\theta = \tan^{-1}(1) = 45°$

12) **When two projectiles are projected with same velocities at complementary angles:**



- When two projectiles are projected with same velocity at two different angles of projection θ and (90- θ) (complementary angles) then they cover same range.

  $\frac{R_1}{R_2} = 1$

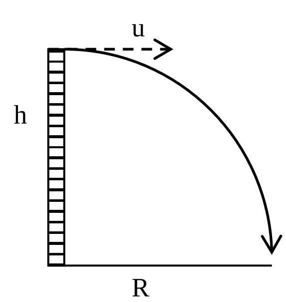
- For a given velocity if T_1 and T_2 are the time of flights for two angles of projection, then $T_1T_2 = \frac{2R}{g}$
- Ratio of time flights $\frac{T_1}{T_2} = \tan\theta$
- Range of the projectile is $R = 4\sqrt{H_1H_2}$ or $H_1H_2 = \frac{R^2}{16}$

  where H_1 and H_2 are the heights reached by the projectiles.

- Ratio of maximum height $\frac{H_1}{H_2} = \tan^2\theta$

  $H_1 + H_2 = \frac{u^2}{2g}$

13) **Projectile projected along horizontal:**



- Velocity of projection $u_x = u$, $u_y = 0$
- Acceleration $a_x = 0$; $a_y = g$
- Horizontal displacement $x = u_x t$ or $t = \frac{x}{u}$

  Vertical displacement $y = \frac{1}{2} gt^2 = \frac{1}{2}\frac{gx^2}{u^2}$

- Position vector: $\vec{r} = x\hat{i} + y\hat{j} = ut\hat{i} + \frac{1}{2}gt^2\hat{j}$
- Time of flight $T = \sqrt{\frac{2h}{g}}$ or $h = \frac{1}{2} gT^2$

  Time taken by the projectile to reach the ground does not depend upon the velocity of projection.

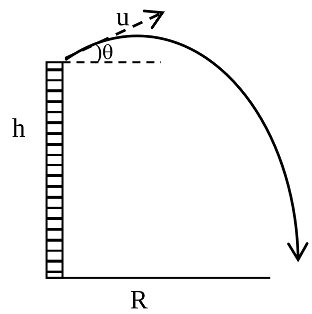
- Horizontal range $R = uT = u\sqrt{\frac{2h}{g}}$
- Instantaneous velocity: $v_x = u$; $v_y = gt = \sqrt{2gh}$

  $v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + g^2t^2} = \sqrt{u^2 + 2gh}$

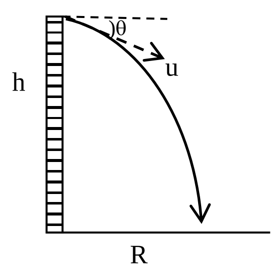
  Direction of velocity $\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

14) **Projection from a height at an angle θ above horizontal:**

- Velocity of projection $u_x = u\cos\theta$, $u_y = -u\sin\theta$
- Acceleration $a_x = 0$; $a_y = g$ (net displacement downward)
- Horizontal displacement $x = (u\cos\theta)t$ or $t = \frac{x}{u\cos\theta}$
- Vertical displacement $y = -(u\sin\theta)t + \frac{1}{2}gt^2$
- Position vector: $\vec{r} = x\hat{i} + y\hat{j}$
- Height: $h = -(u\sin\theta)t + \frac{1}{2}gt^2$
  $h + (u\sin\theta)t - \frac{1}{2}gt^2 = 0$
- Horizontal range $R = (u\cos\theta)t$
- **Velocity after falling height h:**
  $v_y^2 = (-u\sin\theta)^2 + 2gh$
  $v_x = u_x = u\cos\theta$
  $v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + 2gh}$
  Direction of velocity $\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

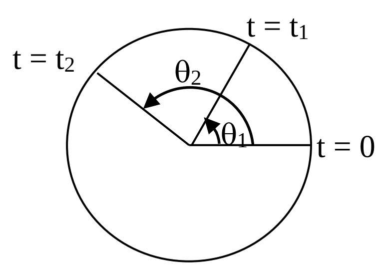


15) **Projection from a height at an angle θ below horizontal:**
- Velocity of projection $u_x = u\cos\theta$, $u_y = u\sin\theta$
- Acceleration $a_x = 0$; $a_y = g$
- Horizontal displacement $x = (u\cos\theta)t$ or $t = \frac{x}{u\cos\theta}$
- Vertical displacement $y = (u\sin\theta)t + \frac{1}{2}gt^2$
- Position vector: $\vec{r} = x\hat{i} + y\hat{j}$
- Height: $h = (u\sin\theta)t + \frac{1}{2}gt^2$
  $h - (u\sin\theta)t - \frac{1}{2}gt^2 = 0$
- Horizontal range $R = (u\cos\theta)t$
- **Velocity after falling height h:**
  $v_y^2 = (u\sin\theta)^2 + 2gh$
  $v_x = u_x = u\cos\theta$
  $v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + 2gh}$
  Direction of velocity $\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

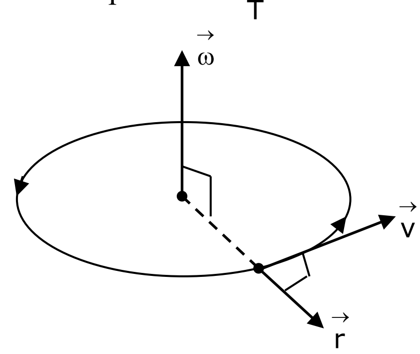


Uniform circular motion

1) Angular displacement of particle $d\theta = ds/r$
where ds is the distance covered and r is the radius of circular path
2) Vector form $\vec{ds} = \vec{d\theta} \times \vec{r}$
3) If $d\theta$ is the angular displacement in time dt then instantaneous angular velocity
$\omega = \frac{d\theta}{dt}$ (rads^{-1})
4) Average angular velocity $\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$
where θ_1 and θ_2 are the angular positions of the particle at instants t_1 and t_2



5) The number of revolutions made per second is called frequency of revolution,
$f = \frac{1}{T}$
6) If 'r' is the radius of the path and T is the period of revolution then linear speed $v = \frac{2\pi r}{T} = 2\pi fr$
7) Relation between linear velocity and angular velocity:
$v = r\omega$
Vector form $\vec{v} = \vec{\omega} \times \vec{r}$
$\vec{v}$ = linear velocity (tangential); $\vec{\omega}$ = angular velocity (along axis)
$\vec{r}$ = radius vector (along radius directed away from centre)
Here $\vec{v} \perp \vec{\omega} \perp \vec{r} \Rightarrow \vec{v}.\vec{\omega} = 0$; $\vec{v}.\vec{r} = 0$; $\vec{\omega}.\vec{r} = 0$



8) Angular velocity $\omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi f$

9) Centripetal acceleration $a = \omega^2 r = \frac{v^2}{r} = v\omega$

10) The centripetal acceleration produced in a particle is because of the centripetal force acting on it.

Centripetal force $F = m\omega^2 r = \frac{mv^2}{r} = mv\omega$

where m is the mass of the particle.

11) Direction of velocity and displacement are along the tangent to the circular path.

12) The centripetal acceleration and centripetal force are directed towards the centre along the radius of the circular path.

13) The velocity and acceleration vectors are perpendicular to each other.

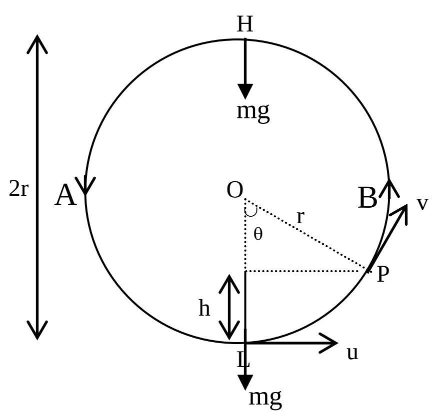
14) Work done by the centripetal force W = Fs cos90 = 0. ($\because$ F and s are perpendicular to each other)

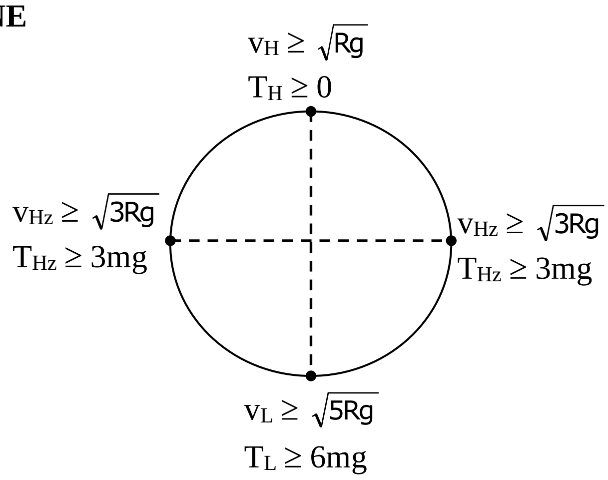
15) Tangential acceleration $a_T = 0$ hence angular acceleration $\alpha = 0$

16) The quantities, which remain constant in uniform circular motion, are speed v, kinetic energy K, angular velocity ω and angular momentum L.

The quantities that changes are velocity v, momentum p, radial (or centripetal) acceleration a_R and centripetal force F_c.

MOTION OF A BODY IN VERTICAL PLANE





Consider a body of mass m tied to one end of a string of length r is rotated in a vertical circle of radius r. It is imparted a velocity u in the horizontal direction at lowest point L.

1) Tension at any position P of angular displacement (θ) along vertical circle is

$T = \frac{mv^2}{r} + mg\cos\theta$

2) Tension in the string is maximum when the body is at the lowest point L ($\theta = 0^o$)

$T_L = \frac{mv_L^2}{r} + mg$

3) Tension in the string is minimum when the body is at the highest point H ($\theta = 180^o$)

$T_H = \frac{mv_H^2}{r} - mg$

4) Body oscillates in lower half circle between A and B if $0 < u \leq \sqrt{3gr}$

5) Body moves to upper half circle but not able to complete loop if $\sqrt{3gr} < u < \sqrt{5gr}$

After leaving the circle, the particle will follow a parabolic path.

6) **For circling the loop**

a) Velocity required at the highest point H is $v_H \geq \sqrt{gr}$

Tension in the string is $T_H \geq 0$

b) When the string is horizontal ($\theta = 90^o$) velocity is $v_{Hz} \geq \sqrt{3gr}$

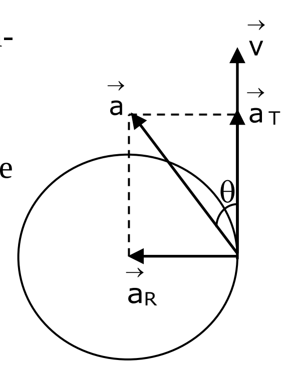
Tension in the string is $T_{Hz} \geq 3mg$

c) Velocity required at the lowest point L is $v_L \geq \sqrt{5gr}$

Tension in the string is $T_L \geq 6mg$

NON-UNIFORM CIRCULAR MOTION:

If a body moves with variable speed in a circle, then the motion is called non-uniform circular motion.



1) In this motion acceleration 'a' has two components:

Centripetal or Radial acceleration $a_R = v\omega = \omega^2 r = \frac{v^2}{r}$ responsible for change in direction only

Tangential acceleration $a_T = r\alpha$ responsible for change in speed only

2) Net acceleration $\vec{a} = \vec{a}_R + \vec{a}_T$ or $|\vec{a}| = \sqrt{a_R^2 + a_T^2}$

3) Net force $\vec{F} = \vec{F}_R + \vec{F}_T$ or $|\vec{F}| = \sqrt{F_R^2 + F_T^2}$

4) As speed v is variable, kinetic energy $K = ½ mv^2$ changes

5) Work done by centripetal force will be zero but work done by tangential force is not zero.
Work done $W = F_T s$, where s is the distance travelled by the body

6) Angle between velocity and net acceleration, $\tan\theta = \frac{a_R}{a_T}$